Determinants of Hankel Matrices

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June 8, 2000

Abstract

The purpose of this paper is to compute asymptotically Hankel determinants for weights that are supported in a semi-infinite interval. The main idea is to reduce the problem to determinants of other operators whose determinant asymptotics are well known.

1 Introduction

The main purpose of this paper is to compute asymptotically the determinants of the finite matrices $H_n(u)$ defined by

$$\det(a_{i+j})_{i,j=0}^{n-1}$$

where

$$a_{i+j} = \int_0^\infty x^{i+j} u(x)\, dx$$

*Supported in part by NSF Grants DMS-9623278 and DMS-9970879.
†Supported in part by NSF Grant DMS-9732687

with suitable conditions on the weight function $u(x)$. These determinant entries depend only the sum $i+j$ and are hence classified as Hankel determinants.

Hankel determinants such as these were considered by Szegö in [10] and also by Hirshmann in [8], but in both cases for finite intervals. These determinants are important in random matrix theory and its applications.

Our main result is as follows. Suppose we replace $u(x)$ by a function given in the form $w(x)U(x)$ where $w(x)$ is the weight $e^{-x}x^{\nu}$ with $\nu \geq -1/2$. Then if U is nowhere zero and $U-1$ is a Schwartz function (a condition which can be considerably relaxed) the determinants are given asymptotically as $n \to \infty$ by

$$\det(H_n(u)) = \exp\{c_1 n^2 \log n + c_2 n^2 + c_3 n \log n + c_4 n + c_5 n^{1/2} + c_6 \log n + c_7 + o(1)\} \quad (1)$$

where

$$c_1 = 1, \quad c_2 = -3/2, \quad c_3 = \nu,$$

$$c_4 = -\nu + \log 2\pi, \quad c_5 = \frac{2}{\pi}\int_0^\infty \log(U(x^2))dx$$

$$c_6 = \nu^2/2 - 1/6, \quad c_7 = 4/3 \log G(1/2) + (1/3 + \nu/2)\log \pi + (\nu/2 - 1/18)\log 2$$

$$- \log G(1+\nu) - \nu/2 \log U(0) + \frac{1}{2\pi^2}\int_0^\infty x S(x)^2 dx,$$

G is the Barnes G-function, and $S(x) = \int_0^\infty \cos(xy)\log U(y^2)dy$.

The idea behind the proof is to replace the matrix $H_n(u)$ with one whose i, j^{th} entry is given by

$$\int_0^\infty P_i(x)P_j(x)u(x)\ dx$$

where the P_i's are orthogonal (Laguerre) polynomials with respect to the weight $e^{-x}x^{\nu}$.

These new determinants can then be evaluated using the ideas of the "linear statistic" method in random matrix theory. More precisely, the above determinant can be replaced by

$$\det(I + K_n)$$

where K_n is an integral operator whose kernel is given by

$$\sum_{i=0}^{n-1} L_i^{\nu}(x)L_i^{\nu}(y)(U-1)(y)$$

with L_i^{ν} defined as the i^{th} Laguerre function of order ν. The main computation in the paper is to approximate the above kernel with a different kernel which involves Bessel functions. This new kernel was fortunately already considered in [2]. There, asymptotics for certain integral operators were computed and these results are then applied to give the result of formula (1). The kernels considered in [2] arises in random matrix theory in the "hard-edge" scaling for ensembles of positive Hermitian matrices. Details about the random matrix connections can be found in [2].

The formula in (1) was stated earlier in [3] where a heuristic argument using the Coulomb gas approach was used to derive the same formula. The Coulomb gas approach was also used in [3] to extend to the case where the interval of integration is the entire real line

2 Preliminaries

We first show how to replace the powers in the Hankel determinants with the orthogonal polynomials. Let

$$\mathcal{H}_n(u) = \left(\int_0^\infty P_i(x)P_j(x)u(x)\ dx\right)\Big|_{i,j=0}^{n-1},$$

and write

$$P_i(x) = \sum_{k\le i} a_{ik}x^k.$$

Lemma 2.1 *Let $P_i, \mathcal{H}_n(u), H_n(u)$ be defined as above and let us assume that all moment integrals exist. Then*

$$\det(\mathcal{H}_n(u)) = A_n \det(H_n(u)),$$

where

$$A_n = (\prod_{i=0}^{n-1} a_{ii})^2.$$

Proof. We have

$$\det(\mathcal{H}_n(u)) = \det(\int_0^\infty (\sum_{m\le j} a_{jm}x^m)(\sum_{l\le k} a_{kl}x^l)u(x)\ dx).$$

The j,k entry of the matrix is given by

$$\sum_{m\le j}\sum_{l\le k} a_{jm}a_{kl}\int_0^\infty x^{m+l}u(x)\ dx$$

$$= \sum_{m\le j}\sum_{l\le k} a_{jm}\int_0^\infty x^{m+l}u(x)\ dx\ a_{kl}.$$

Form this it follows that $\mathcal{H}_n(u)$ is a product of three matrices $TH_n(u)T^t$. The matrix T is lower triangular with entries $a_{jm}, m \le j$. It is easy to see from this that the lemma follows.

The next step is to evaluate the term

$$A_n = (\prod_{i=0}^{n-1} a_{ii})^2,$$

which is of course straight forward. We normalize the polynomials so that they are orthonormal. Then it is well known [9] that

$$a_{ii}^2 = \frac{1}{\Gamma(1+i+\nu)\Gamma(1+i)}.$$

The following lemma computes the product asymptotically using the Barnes G-function. This function is defined by [1, 12]

$$G(1+z) \quad = \quad (2\pi)^{z/2}e^{-(z+1)z/2-\gamma_E z^2/2}\prod_{k=1}^{\infty}\left((1+z/k)^k e^{-z+z^2/2k}\right) \qquad (2)$$

with γ_E being Euler's constant.

Lemma 2.2

$$A_n^{-1} = \prod_{i=0}^{n-1} a_{ii}^{-2}$$

is given asymptotically by

$$\exp\left\{d_1 n^2 \log n + d_2 n^2 + d_3 n \log n + d_4 n + d_5 \log n + d_6 + o(1)\right\}$$

where

$$d_1 = 1,\ \ d_2 = -3/2,\ \ d_3 = \nu,\ \ d_4 = -\nu + \log 2\pi,\ \ d_5 = \nu^2/2 - 1/6$$

$$d_6 = 4/3 \log G(1/2) + (1/3 + \nu/2) \log \pi + (\nu/2 - 1/18) \log 2 - \log G(1+\nu)$$

Proof. It is well known that the Barnes G function satisfies the property [12]

$$G(1+z) = \Gamma(z) G(z).$$

From this it is quite easy to see that A_n^{-1} can be written as

$$\frac{G(1+n)G(1+n+\nu)}{G(1+\nu)}.$$

The asymptotics of the Barnes function are computed in [12] and since $G(1+a+n)$ is asymptotic to

$$n^{(n+a)^2/2-1/12} e^{-3/4n^2-an} (2\pi)^{(n+a)/2} G^{2/3}(1/2) \pi^{1/6} 2^{-1/36}$$

we can directly apply this formula with $a = 0$ and $a = \nu$ to obtain the desired result.

This last result shows then that the asymptotics of the Hankel matrices can be reduced to those of the matrices $\mathcal{H}_n(u)$. We compute these asymptotics by replacing the determinant with a Fredholm determinant. To do this we must first consider some estimates for the various kernels.

3 Hilbert-Schmidt and trace norm estimates

We proceed to write the determinant of $\mathcal{H}_n(u)$ as a Fredholm determinant. It is clear that if $U(x)$ is a bounded function then the matrix $\mathcal{H}_n(u)$ can be realized as $P_n M_U P_n$ where M_U is multiplication by U and P_n is the projection onto the space spanned by the first n Laguerre functions. This is clearly a bounded, finite rank operator defined on $L_2(0,\infty)$. Thus

$$\det \mathcal{H}_n(u) = \det(I + K_n)$$

is defined where K_n is the integral operator whose kernel is given by

$$\sum_{i=0}^{n-1} L_i^\nu(x) L_i^\nu(y)(U(y) - 1).$$

Let us write a square root of the function $U-1$ as V. Then

$$\det(I+K_n) = \det(I+P_nM_VM_VP_n) = \det(I+M_VP_nM_V).$$

This last equality uses the general fact that

$$\det(I+AB) = \det(I+BA)$$

for general operators defined on a Hilbert space. At this point we have replaced the kernel K_n with

$$\sum_{i=0}^{n-1} V(x)L_i^\nu(x)L_i^\nu(y)V(y).$$

Our primary goal in the paper is to replace this last kernel, which we will commonly refer to as the Laguerre kernel, with a more familiar one namely the compressed Bessel kernel, also defined on $L_2(0,\infty)$,

$$V(x)\frac{J_\nu(2(nx)^{1/2})\sqrt{ny}J_\nu'(2(ny)^{1/2}) - J_\nu(2(ny)^{1/2})\sqrt{nx}J_\nu'(2(nx)^{1/2})}{x-y}V(y). \tag{3}$$

We use the term "compressed" since the above without the factors $V(x)$ and $V(y)$ is generally called the Bessel kernel.

We will use the general fact that if we have families of trace class operators A_n and B_n (thinking of the Laguerre kernel as A_n and the compressed Bessel kernel as B_n) that depend on a parameter n such that the Hilbert Schmidt norm $||A_n - B_n||_2 = o(1)$ and $\mathrm{tr}\,(A_n-B_n)(I+B_n)^{-1} = o(1)$ then

$$\det(I+A_n)/\det(I+B_n) \ \to 1$$

as $n\to\infty$. The proof of this fact uses the idea of generalized determinants found in [7], and also requires uniformity for the norms of the inverses of the operators $I+B_n$. This will be made more precise later.

Using the standard identity for the sum of the products of Laguerre functions allows us to write the Laguerre kernel

$$\sum_{i=0}^{n-1} V(x)L_i^\nu(x)L_i^\nu(y)V(y)$$

as

$$\sqrt{n(n+\nu)}V(x)\frac{L_{n-1}^\nu(x)L_n^\nu(y) - L_{n-1}^\nu(y)L_n^\nu(x)}{x-y}V(y).$$

Notice that the above kernel has somewhat the same form as the compressed Bessel kernel. It turns out that both the kernel above and the Bessel kernel have an integral representation that make the Hilbert-Schmidt computations simpler. The next lemma shows how this is done for in the Laguerre case.

Lemma 3.1 *We have*

$$\frac{L_{n-1}^{\nu}(x)L_n^{\nu}(y)-L_{n-1}^{\nu}(y)L_n^{\nu}(x)}{x-y}=\frac{1}{2}\int_0^1 [L_{n-1}^{\nu}(tx)L_n^{\nu}(ty)+L_{n-1}^{\nu}(ty)L_n^{\nu}(tx)]\,dt.$$

Proof. Call the left-hand side $\Phi(x,y)$ and the right-hand side $\Psi(x,y)$. It follows from the differentiation formulas for Laguerre polynomials that there is a differentiation formula

$$x\frac{d}{dx}\begin{pmatrix} L_{n-1}^{\nu}(x) \\ L_n^{\nu}(x)\end{pmatrix}=\begin{pmatrix} A & B \\ -C & -A\end{pmatrix}\begin{pmatrix} L_{n-1}^{\nu}(x) \\ L_n^{\nu}(x)\end{pmatrix},$$

where

$$A(x)=\frac{1}{2}x-\frac{\nu}{2}-N,\quad B(x)=C(x)=\sqrt{N(N+\nu)}\,.$$

An easy computation using this shows that $\Phi(x,y)$ satisfies

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+1\right)\Phi(x,y)=\frac{1}{2}[L_{n-1}^{\nu}(x)L_n^{\nu}(y)+L_{n-1}^{\nu}(y)L_n^{\nu}(x)].$$

On the other hand, if we temporarily assume that $\nu>0$ and differentiate under the integral sign and then integrate by parts we find that

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+1\right)\Psi(x,y)=\frac{1}{2}[L_{n-1}^{\nu}(x)L_n^{\nu}(y)+L_{n-1}^{\nu}(y)L_n^{\nu}(x)].$$

Hence

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+1\right)(\Phi(x,y)-\Psi(x,y))=0.$$

From this it follows that $\Phi(x,y)-\Psi(x,y)$ is of the form $\varphi(x/y)\,e^{-(x+y)/2}$ for some function φ of one variable. Still assuming $\nu>0$, both $\Phi(x,y)$ and $\Psi(x,y)$ tend to 0 as x and y tend to 0 independently. This shows that $\varphi=0$ and so the identity is established when $\nu>0$. Since both sides of the identity are analytic functions of ν for Re $\nu>-1$ the identity holds generally.

Thus our kernel may be rewritten

$$\frac{1}{2}\sqrt{n(n+\nu)}\,V(x)V(y)\int_0^1 [L_{n-1}^{\nu}(tx)L_n^{\nu}(ty)+L_{n-1}^{\nu}(ty)L_n^{\nu}(tx)]\,dt. \tag{4}$$

We assume from now on that $|\nu|\geq 1/2$. The compressed Bessel kernel appearing in (3) also has the following well-known integral representation

$$nV(x)V(y)\int_0^1 J_\nu(2\sqrt{nxt})J_\nu(2\sqrt{nyt})dt \tag{5}$$

and the rest of the section is devoted to replacing the Laguerre functions by the appropriate Bessel functions in (4) and showing that this leads to a small Hilbert-Schmidt error and then showing that certain traces tend to zero. Notice that in the above notation kernel (4) is the same as A_n and kernel (5) is the same as B_n. We need only the following fact about the Laguerre functions.

Lemma 3.2 *Let a be any real positive constant and suppose that $\nu \geq -1/2$. Then as $n \to \infty$, the normalized Laguerre functions satisfy*

$$\max_{0<x\leq a} x^{1/4} L_n^\nu(x) = O(n^{-1/4}),$$

$$\max_{x\geq a} L_n^\nu(x) = O(n^{-1/4}).$$

We remark that the implied constants in the estimates may depend on a but not on n.

Proof. These estimates follow easily from formula (7.6.9) and Theorem 8.91.2 in [9].

Lemma 3.3 *We have*

$$L_n^\nu(x) - c_{n,\nu} J_\nu(2\sqrt{Nx}) = \frac{c_{n,\nu}}{6\sqrt{\pi}} N^{-3/4} x^{5/4} \sin(2\sqrt{Nx} - \alpha) + O(N^{-5/4}(x^{-1/4} + x^3)), \qquad (6)$$

where $N = n + (\nu+1)/2, c_{n,\nu} = N^{-\nu/2}(\frac{\Gamma(n+\nu+1)}{\Gamma(n+1)})^{1/2}$, $\alpha = (\nu/2 + 1/4)\pi$ and the constant implied in the O depends only on ν.

Proof. Formula (8.64.3) of Szegö reads, in current notation,

$$L_n^\nu(x) - c_{n,\nu} J_\nu(2\sqrt{Nx})$$

$$= \frac{\pi}{4\sin\nu\pi} \int_0^x [J_\nu(2\sqrt{Nx}) J_{-\nu}(2\sqrt{Nt}) - J_{-\nu}(2\sqrt{Nx}) J_\nu(2\sqrt{Nt})]\, t\, L_n^\nu(t)\, dt.$$

Using the Lemma 3.2 estimate for $L_n^\nu(t)$ we find that if $Nx < 1$ the right side is at most a constant times

$$(Nx)^{-|\nu|/2} \int_0^x (Nt)^{-|\nu|/2}\, t\, L_n^\nu(t)\, dt = O(N^{-|\nu|-\frac{1}{4}} x^{\frac{7}{4}-|\nu|}) = O(N^{-2}).$$

(If ν is an integer this must be multiplied by $\log N$.)

So we assume $Nx > 1$. The integral over $Nt < 1$ is at most a constant times

$$(Nx)^{-1/4} \int_0^{1/N} (Nt)^{-|\nu|/2} N^{-1/4} t^{\frac{3}{4}}\, dt = O(N^{-\frac{9}{4}} x^{-1/4})$$

with, possibly, an extra $\log N$ factor. So we confine our integral now to $Nt > 1$.

If we replace the Bessel functions by their first-order asymptotics the error is at most a constant times

$$N^{-1} \int_0^x (x^{-1/2} + t^{-1/2})(xt)^{-1/4} N^{-1/4}(t + t^{3/4})\, dt = O(N^{-5/4}(1+x)).$$

Therefore with this error we can replace the Bessel functions by their first-order asymptotics, obtaining, after using some some trigonometric identities,

$$\frac{1}{4} N^{-1/2} x^{-1/4} \int_{1/N}^x \sin(2\sqrt{Nx} - 2\sqrt{Nt})\, t^{3/4} L_n^\nu(t)\, dt. \qquad (7)$$

Notice that this has the uniform estimate $O(N^{-3/4}(1+x^{3/2}))$, and the earlier errors combined are $O(N^{-5/4}(x^{-1/4}+x)$. It follows that if in this integral we replace $L_n^\nu(t)$ by $c_{n,\nu}J_\nu(2\sqrt{Nt})$ the error is $O(N^{-5/4}(1+x^3))$. Then if we replace $J_\nu(2\sqrt{Nt})$ by its first-order asymptotics the error is at most a constant times

$$N^{-1/2}x^{-1/4}\int_{1/N}^{x} t^{3/4}\,(Nt)^{-3/4}\,dt = O(N^{-5/4}x^{3/4}).$$

Hence with the sum of the last-mentioned errors we may replace the factor $L_n^\nu(t)$ in (7) by the first-order asymptotics of $c_{n,\nu}J_\nu(2\sqrt{Nt})$. Using a trigonometric identity shows that this results in

$$\frac{c_{n,\nu}}{8\sqrt{\pi}}N^{-3/4}x^{-1/4}\int_{1/N}^{x}[\sin(2\sqrt{Nx}-\alpha)-\sin(2\sqrt{Nx}-4\sqrt{Nt}+\alpha)]\,t^{1/2}\,dt$$

$$=\frac{c_{n,\nu}}{6\sqrt{\pi}}N^{-3/4}x^{5/4}\sin(2\sqrt{Nx}-\alpha)+O(N^{-9/4}x^{-1/4})+O(N^{-5/4}x^{3/4}).$$

Putting these things together gives the statement of the lemma.

In what follows we use the notation $o_2(\,\cdot\,)$ or $O_2(\,\cdot\,)$ for a family of operators whose Hilbert-Schmidt norm satisfies the corresponding o or O estimate, or for a kernel whose associated operator satisfies the estimate. Similarly, $o_1(\,\cdot\,)$ and $O_1(\,\cdot\,)$ refer to the trace norm. We also set $N' = n+(\nu-1)/2$.

Lemma 3.4 . *Suppose V is in L^∞ and satisfies $\int_0^\infty |V(x)|^2(x^{-1/2}+x^6)dx<\infty$. Then the difference between the kernel (4) and*

$$\frac{1}{2}\sqrt{n(n+\nu)}V(x)V(y)c_{n-1,\nu}c_{n,\nu}\int_0^1[J_\nu(2\sqrt{N'tx})J_\nu(2\sqrt{Nty})+J_\nu(2\sqrt{N'ty})J_\nu(2\sqrt{Ntx})]\,dt$$

is equal to
(i) $O_1(N^{-1/2})$ plus a constant which is $O(1)$ times

$$N^{1/4}\int_0^1\left[(tx)^{5/4}\sin(2\sqrt{N'tx}-\alpha)J_\nu(2\sqrt{Nty})+(ty)^{5/4}\sin(2\sqrt{Nty}-\alpha)J_\nu(2\sqrt{N'tx})\right]V(x)V(y)\,dt$$

plus a similar term with x and y interchanged;
(ii) $O_2(N^{-1/4})$.

Proof. The estimate of Lemma 3.3 holds with n replaced by $n-1$ if N is replaced by N'. Write

$$L_{n-1}^\nu(x)L_n^\nu(y)-c_{n-1,\nu}c_{n,\nu}J_\nu(2\sqrt{N'x})J_\nu(2\sqrt{Ny})$$

$$=[L_{n-1}^\nu(x)-c_{n-1,\nu}J_\nu(2\sqrt{N'x})]\,L_n^\nu(y)+L_{n-1}^\nu(x)[L_n^\nu(y)-c_{n,\nu}J_\nu(2\sqrt{Nx})]$$

$$+[c_{n-1,\nu}J_\nu(2\sqrt{N'x})-L_{n-1}^\nu(x)][L_n^\nu(y)-c_{n,\nu}J_\nu(2\sqrt{Nx})].$$

The contribution of the right side to the difference of the two kernels is a constant which is $O(1)$ times

$$N\int_0^1 \Big([L_{n-1}^\nu(tx)-c_{n-1,\nu}J_\nu(2\sqrt{N'tx})]\,L_n^\nu(ty)$$
$$+L_{n-1}^\nu(tx)[L_n^\nu(ty)-c_{n,\nu}J_\nu(2\sqrt{Ntx})]$$
$$+\,[c_{n-1,\nu}J_\nu(2\sqrt{N'tx})-L_{n-1}^\nu(tx)][L_n^\nu(ty)-c_{n,\nu}J_\nu(2\sqrt{Ntx})]\Big)\,V(x)V(y)\,dt.$$

The integrand is a sum of three terms, each of which is the kernel (in the x, y variables) of a rank one operator. The trace norm of such an integral is at most the integral of the Hilbert-Schmidt norms.) Using this fact and Lemma 3.2 we see that the contribution of the error term in (6) to any of these terms is $O_1(N^{-1/2})$, as is the contribution of the last term above. Then we see that replacing the two Laguerre functions by the corresponding Bessel functions leads to an even smaller error. The error term in Lemma 3.3 is seen also to contribute $O_1(N^{-1/2})$. Applying this lemma, and then doing everything with x and y interchanged we arrive at the statement of part (i).

For part (ii), we have to show that the integrals are $O_2(N^{-1/2})$. It is easy to see that with error $O_1(N^{-1/2})$ we may replace the Bessel functions by their first-order asymptotics, resulting in a constant which is $O(N^{-1/4})$ times

$$V(x)V(y)x^{5/4}y^{-1/4}\int_0^1 \sin(2\sqrt{N'tx}-\alpha)\,\cos(2\sqrt{Nty}-\alpha)\,t\,dt$$

plus similar expressions. A trigonometric identity and integration by parts shows that the integral is at most a constant times

$$(\max(1,\,|2\sqrt{N'x}-2\sqrt{Ny}|))^{-1}+(\max(1,\,|2\sqrt{N'x}+2\sqrt{Ny}|-2\alpha))^{-1},$$

and an easy exercise shows that, together with the outer factors, this gives $O_2(N^{-1/4})$. Analogous argument applies to the other integrals and this completes the proof of the lemma.

Lemma 3.5 *The kernel (4) is equal to*

$$n\,V(x)V(y)\int_0^1 J_\nu(2\sqrt{ntx})J_\nu(2\sqrt{nty})\,dt \tag{8}$$

plus an error $o_2(1)$.

Proof. Since the Laguerre kernel has Hilbert-Schmidt norm $n^{1/2}$ (the operator is a rank n projection) multiplying (4) by constants which are $1+O(n^{-1})$ produces an error $O_2(n^{-1/2})$. The constants we choose are $n/(\sqrt{n(n+\nu)}c_{n-1,\nu}c_{n,\nu})$. It follows from this and Lemma 3.4 that with error $o_2(1)$ we can replace (4) by

$$\frac{n}{2}V(x)V(y)\int_0^1[J_\nu(2\sqrt{N'tx})J_\nu(2\sqrt{Nty})+J_\nu(2\sqrt{N'ty})J_\nu(2\sqrt{Ntx})]\,dt. \tag{9}$$

Now we show that if we replace N and N' by n in this kernel the error is $o_2(1)$. Let's look at the error incurred in the integral involving the first summand when we replace N' by n. It equals

$$\int_{N'}^{n}\frac{dr}{\sqrt{r}}\int_0^1 \sqrt{tx}\,J_\nu'(2\sqrt{rtx})\,J_\nu(2\sqrt{Nty})\,dt. \tag{10}$$

If $ntx<1$ and $nty>1$ the inner integral is at most a constant times

$$n^{-\mu/2-1/2}x^{-\mu/2}\int_0^{\min(1,\,1/nx)} t^{-\mu/2}dt,$$

where $\mu=\max(-\nu,\,0)$ as before. This is bounded by a constant times

$$n^{-1/2}(nx)^{-\mu/2} \text{ if } nx<1, \qquad n^{-3/2}x^{-1} \text{ if } nx>1.$$

This times $V(x)V(y)$ is $O_2(n^{-1})$ and so its eventual contribution to the Hilbert-Schmidt norm (because of the external factor n and the fact that $r\sim n$) is $O(n^{-1/2})$.

If $ntx<1$ and $nty<1$ the inner integral is at most a constant times

$$n^{-\mu-1/2}x^{-\mu/2}y^{-\mu/2}\int_0^{\min(1,\,1/nx,\,1/ny)} t^{-\mu}dt.$$

By symmetry we may assume $y<x$. If $nx<1$ this is at most a constant times the outer factor which, when multiplied by $V(x)V(y)$, is $O_2(n^{-3/2})$. If $nx>1$ the above is at most a constant times

$$n^{-3/2+\mu/2}x^{-1+\mu/2}y^{-\mu/2},$$

and this times $V(x)V(y)$ is $O_2(n^{-1})$. Thus the eventual contribution of the portion of the integral where $ntx<1$ is $O_2(n^{-1/2})$.

In the region $ntx>1$, if we replace the first Bessel function in (10) by the first term of its asymptotic expansion it is easy to see we incur in the end an error $O_2(n^{-1/4})$. After this replacement the inner integral becomes a constant times

$$n^{-1/4}x^{1/4}\int_{1/nx}^{1} t^{1/4}\cos(2\sqrt{rtx}-\alpha)\,J_\nu(2\sqrt{Nty})\,dt$$

$$=n^{-3/2}x^{1/4}\int_{1/x}^{n} t^{1/4}\cos(2\sqrt{rtx/n}-\alpha)\,J_\nu(2\sqrt{ty})\,dt.$$

Taking account of the external factor of n in our kernel, and the r-integral, we see that we want to show that the Hilbert-Schmidt norm of the kernel

$$V(x)V(y)\,n^{-1}\int_{1/x}^{n} t^{1/4}\cos(2\sqrt{rtx/n}-\alpha)\,J_\nu(2\sqrt{ty})\,dt$$

tends to 0 as $n\to\infty$. But from the asymptotics of the Bessel function it is clear that n^{-1} times the integral is uniformly bounded by a constant times $1+y^{-\mu/2}$ and tends to 0 whenever $x\neq y$. Hence the dominated convergence theorem tells us that the product is $o_2(1)$.

An analogous argument shows that replacing N' by n in the second summand of (9) and then replacing N by n in both summands leads to an error $o_2(1)$. This completes the proof.

Remark. In the preceding lemmas our various kernels had the factor $V(x)V(y)$. It is easy to see from their proofs that the lemmas hold with this factor replaced everywhere by $V_1(x)V_2(y)$ as long as V_1 and V_2 are bounded and have sufficiently rapid decay at infinity. The next lemma is the first that will require some smoothness.

We shall denote by $J_n(x,y)$ the kernel (5) without the external factor $V(x)V(y)$, and by J_n the corresponding operator. As before we denote by M_V multiplication by V so that (5) is the kernel of the operator $M_V\, J_n\, M_V$.

Lemma 3.6 *For any Schwartz function W the commutator $[W,\, J_n]$ has Hilbert-Schmidt norm which is bounded as $n \to \infty$.*

Proof. Write the kernel of the commutator as in formula (3). It has the form

$$\frac{W(x)-W(y)}{x-y}(J_\nu(2\sqrt{nx})\sqrt{ny}J_\nu^{'}(2\sqrt{ny})) \tag{11}$$

plus a similar term with x an y interchanged. If $nx > 1$ and $ny > 1$ then the product

$$J_\nu(2\sqrt{nx})\sqrt{ny}J_\nu^{'}(2\sqrt{ny})$$

is $O(x^{-1/4}y^{1/4})$, and thus if $|x-y|$ is bounded away from zero, or if we integrate over any bounded region the Hilbert-Schmidt norm (the square root of the integral of the square) is bounded. If $|y-x| < 1$, and assuming $x, y > 1$ we see that y/x is bounded and thus the Hilbert-Schmidt norm can be estimated by the square root of

$$\int_1^\infty \int_1^\infty |\frac{W(x)-W(y)}{x-y}|^2\, dx\, dy.$$

But this is known [11] to be bounded by $\int_{-\infty}^{\infty}|x||\hat{W}(x)|^2dx$. Now suppose that $nx < 1$ and $ny > 1$. Then the product of Bessel functions is $O(n^{-\mu+1/4}x^{-\mu/2}y^{1/4})$. If $|x-y|$ is bounded away from zero, then the resulting Hilbert-Schmidt norm is $O(n^{-1/4})$. If $|x-y| < 1$, then it is also clear that the Hilbert-Schmidt norm is $O(n^{-1/4})$. The other two cases $xn > 1, yn < 1$ and $xn < 1, yn < 1$ are handled in the same fashion and are left to the reader.

Lemma 3.7 *For any bounded functions W_1 and W_2 we have*

$$\mathrm{tr}\ (A_n - B_n)\, W_2\, J_n\, W_1 \to 0$$

as $n \to \infty$.

Proof. For convenience all kernels $K(x,y)$ in this proof will be replaced by their unitary equivalents $2\sqrt{xy}K(x^2,\,y^2)$. We denote by $L_n(x,y)$ this unitary equivalent of the Laguerre kernel (4) without the external V factors and by $J_n(x,y)$ here the unitary equivalent of the Bessel kernel (5) without the V factors. We also make the substitution $t\to t^2$ in the t integrals. Thus in our present notation

$$J_n(x,y)=4n\sqrt{xy}\int_0^1 J_\nu(2\sqrt{n}xt)\,J_\nu(2\sqrt{n}yt)\,tdt. \tag{12}$$

We also denote by $\tilde{J}_n(x,y)$ the unitary equivalent of the first displayed operator of Lemma 3.4, without the V factors. Thus

$$\tilde{J}_n(x,y)=2\sqrt{n(n+\nu)}c_{n-1,\nu}c_{n,\nu}$$

$$\times\sqrt{xy}\int_0^1 J_\nu[(2\sqrt{N'}tx)J_\nu(2\sqrt{N}ty)+J_\nu(2\sqrt{N'}ty)J_\nu(2\sqrt{N}tx)]tdt. \tag{13}$$

If we set $V_i(x)=V(x^2)W_i(x^2)$ then we see that our trace equals $\mathrm{tr}\,M_{V_1}(L_n-J_n)M_{V_2}J_n$. We shall show that this goes to 0 in two steps, showing first that $\mathrm{tr}\,M_{V_1}(L_n-\tilde{J}_n)M_{V_2}J_n\to 0$ and then that $\mathrm{tr}\,M_{V_1}(\tilde{J}_n-J_n)M_{V_2}\to 0$.

First, the asymptotics of the Bessel functions gives

$$J_\nu(z)=\sqrt{\frac{2}{\pi z}}\cos(z-\alpha)+O\left(\frac{z^{-1/2}}{<z>}\right),$$

where $<z>=(1+z^2)^{1/2}$. (This also uses $\nu\ge -1/2$.) Hence

$$J_n(x,y)=\frac{8}{\pi}n\sqrt{xy}\int_0^1\left[\frac{\cos(2\sqrt{n}tx-\alpha)}{(2\sqrt{n}tx)^{1/2}}+O\left(\frac{(\sqrt{n}tx)^{-1/2}}{<(\sqrt{n}tx)>}\right)\right]$$

$$\times\left[\frac{\cos(2\sqrt{n}ty-\alpha)}{(2\sqrt{n}ty)^{1/2}}+O\left(\frac{\sqrt{n}ty)^{-1/2}}{<(\sqrt{n}ty)>}\right)\right]tdt$$

$$=\frac{8}{\pi}n\int_0^1\cos(2\sqrt{n}tx-\alpha)\,\cos(2\sqrt{n}ty-\alpha)dt$$

$$+O\left(\frac{\sqrt{n}}{<\sqrt{n}x>}+\frac{\sqrt{n}}{<\sqrt{n}x>}+\frac{\sqrt{n}}{<\sqrt{n}x>^{1/2}<\sqrt{n}y>^{1/2}}\right).$$

The last summand is at most a constant times the sum of the preceding two. Using this, a trigonometric identity and integrating we find that

$$J_n(x,y)=O\left(\frac{\sqrt{n}}{<\sqrt{n}(x-y)>}+\frac{\sqrt{n}}{<\sqrt{n}(x+y)>}+\frac{\sqrt{n}}{<\sqrt{n}x>}+\frac{\sqrt{n}}{<\sqrt{n}y>}\right). \tag{14}$$

We consider first $M_{V_1}(L_n-\tilde{J}_n)M_{V_2}J_n$. Lemma 3.4 tells us that the kernel of $M_{V_1}(L_n-\tilde{J}_n)M_{V_2}$ is $O_1(n^{-1/2})$ plus the unitary equivalent of the expression in part (i) with modified

V factors. (See the remark following Lemma 3.5.) In the proof of part (ii) it was stated that if we replace the Bessel functions in this expression by their first order asymptotics the error is $O_1(n^{-1/2})$. (We shall go through the details for similar integrals below.) So we may replace $L_n - \tilde{J}_n$ by a constant which is $O(1)$ times

$$\int_0^1 [x^3 \sin(2\sqrt{N'}tx - \alpha) \cos(2\sqrt{N}ty - \alpha) + y^3 \sin(2\sqrt{N'}ty\alpha) \cos(2\sqrt{N}tx - \alpha)]\, t^3 dt.$$

(Recall the unitary equivalents we are using and the variable change $t \to t^2$.) If in the integrals we made the replacements $N,\ N' \to n$ we would incur an error $O((x^4 + y^4)/\sqrt{n})$. Multiplying this by $V_1(x)V_2(y)$ times (14) and integrating is easily seen to give $o(1)$. After these replacements the integral becomes what may be written

$$(x^3 - y^3) \int_0^1 \sin(2\sqrt{n}tx - \alpha) \cos(2\sqrt{n}ty - \alpha)\, t^3 dt + y^3 \int_0^1 \sin(2\sqrt{n}t(x+y) - 2\alpha)\, t^3 dt$$

$$= O\left(\frac{x^3 - y^3}{< \sqrt{n}(x-y) >} + \frac{x^3 - y^3}{< \sqrt{n}(x+y) >} + \frac{y^3}{< \sqrt{n}(x+y) >}\right)$$

$$= O\left(\frac{x^3 - y^3}{< \sqrt{n}(x-y) >} + \frac{x^3 + y^3}{< \sqrt{n}(x+y) >}\right).$$

Let us see why if we multiply this by $V_1(x)V_2(y)$ times (14) and integrate we get $o(1)$.

First,

$$\frac{x^3 - y^3}{< \sqrt{n}(x-y >} \frac{\sqrt{n}}{< \sqrt{n}(x-y) >} = O\left(\frac{x^2 + y^2}{< \sqrt{n}(x-y) >}\right)$$

goes to zero pointwise and this times $V_1(x)V_2(y)$ is bounded by a fixed L^1 function. Thus the integral of the product goes to zero. The term with $< \sqrt{n}(x+y) >$ instead of $< \sqrt{n}(x-y) >$ is even smaller.

Next consider

$$\frac{x^3 - y^3}{< \sqrt{n}(x-y) >} \frac{\sqrt{n}}{< \sqrt{n}x >}.$$

We may ignore the factor $x^3 - y^3$ since it may be incorporated into the V_i. After the substitutions $x \to x/\sqrt{n},\ y \to y/\sqrt{n}$ the integral in question becomes

$$\frac{1}{\sqrt{n}} \int\int \frac{|V_1(x/\sqrt{n})V_2(y/\sqrt{n})|}{< x - y >< x >} dy dx.$$

Schwarz's inequality shows that the y integral is $O(n^{1/4})$ so our double integral is bounded by a constant times

$$n^{-1/4} \int_0^1 dx + n^{-1/4} \int_1^\infty \frac{|V_1(x/\sqrt{n})|}{x} dx = n^{-1/4} + n^{-1/4} \int_{1/\sqrt{n}}^\infty \frac{|V_1(x)|}{x} dx = O(n^{-1/4} \log n).$$

Again the term with $< \sqrt{n}(x+y) >$ instead of $< \sqrt{n}(x-y) >$ is even smaller.

Now we look at $M_{V_1}(\tilde{J}_n - J_n)M_{V_2}J_n$. To find bounds for $\tilde{J}_n(x,y) - J_n(x,y)$ let us look first at the error incurred if in the first integral in (13) we replace N' by n. The error in the integral together with the external factor $\sqrt{xy}$ equals

$$\int_{N'}^{n} \frac{dr}{\sqrt{r}} \int_0^1 x^{3/2} y^{1/2} J_\nu'(2\sqrt{r}tx) J_\nu(2\sqrt{n}ty)\, t^2 dt.$$

Using the asymptotics of $J_\nu(z)$ and the fact that

$$J_\nu'(z) = -\sqrt{\frac{2}{\pi z}} \sin(z-\alpha) + O(z^{-3/2}),$$

we can write the above as a constant times

$$\int_{N'}^{n} \frac{dr}{\sqrt{r}} \int_0^1 x^{3/2} y^{1/2} \left[\frac{\sin(2\sqrt{r}tx - \alpha)}{(\sqrt{r}tx)^{1/2}} + O((\sqrt{n}tx)^{-3/2}) \right]$$

$$\times \left[\frac{\cos(2\sqrt{N}ty - \alpha)}{(\sqrt{N}ty)^{1/2}} + O\left(\frac{(\sqrt{n}ty)^{-1/2}}{< (\sqrt{n}ty) >} \right) \right] t^2 dt.$$

We estimate the trace norm of $V_1(x)V_2(y)$ times this by taking the trace norm under the integral signs. Since the integrand is, for fixed r and t, a function of x times a function of y its trace norm equals the product of the L^2 norms of its factors. In multiplying out we will have main terms and error terms and we must estimate norms of all products. Thus we compute (in each line there will be an integral corresponding to a main term and then and error term)

$$\int \frac{x^3 |V_1(x)|^2}{\sqrt{r}tx} dx = O(n^{-1/2}t^{-1}), \quad \int \frac{x^3 |V_1(x)|^2}{(\sqrt{n}tx)^3} dx = O(n^{-3/2}t^{-3}),$$

$$\int \frac{y|V_2(y)|^2}{\sqrt{N}ty} dy = O(n^{-1/2}t^{-1}), \quad \int \frac{y(\sqrt{N}ty)^{-1}|V_2(y)|^2}{< \sqrt{N}ty >^2} dy = O(n^{-1}t^{-2}).$$

Combining L^2 norms we see that the trace norm of the contribution to the integrand of all but the product of the main terms is $O(n^{-3/4}t^{-2})$. Integrating over t we are left with $O(n^{-3/4})$ and integrating over r gives $O(n^{-5/4})$. If we combine this we the external factor in (13) which is $O(n)$ we are left with $O(n^{-1/4})$. Since the operator norms of the J_n are bounded the eventual contribution to the trace of the product will be $O(n^{-1/4})$.

Thus we are left with the main term, which is

$$\int_{N'}^{n} (rN)^{-1/4} \frac{dr}{\sqrt{r}} \int_0^1 x \sin(2\sqrt{r}tx - \alpha)\, \cos(2\sqrt{N}ty - \alpha)\, t dt.$$

If in this we replaced r and N by n everywhere in the integrand the error would be $O(n^{-3/2})$. If we multiply this by $V_1(x)V_2(y)$ and use the estimate (14) we find by dominated convergence

that the product has trace tending to zero, even keeping in mind the extra factor $O(n)$ in (13). So we may make these replacements, which results in

$$(n - N')\, n^{-1} \int_0^1 x \sin(2\sqrt{n}tx - \alpha)\, \cos(2\sqrt{n}ty - \alpha)\, tdt.$$

Now there is a second integral in $\tilde{J}_n$, which is obtained from the first by interchanging x and y. Interchanging and adding gives what can be written

$$(n - N')\, n^{-1} \int_0^1 [(x-y)\sin(2\sqrt{n}tx - \alpha)\, \cos(2\sqrt{n}ty - \alpha) + y\, \sin(2\sqrt{n}tx + 2\sqrt{n}ty - 2\alpha)]\, tdt$$

$$= O\left(\frac{n^{-3/2}(x-y)}{<\sqrt{n}(x-y)>} + \frac{n^{-3/2}(x+y)}{<\sqrt{n}(x+y>}\right).$$

If we multiply by $V_1(x)V_2(y)$ and use the estimate (14) we find again that the product has trace tending to zero, even keeping in mind the extra factor $O(n)$.

Thus replacement of N' by n in (13) leads to an eventual error in the trace of $o(1)$. Similarly so does then the replacement of N by n. Finally,

$$\sqrt{n(n+\nu)}c_{n-1,\nu}c_{n,\nu} = n\,(1 + O(n^{-1})),$$

so the eventual error in the trace upon replacing the constant by n is $O(n^{-1})$ times what is obtained by multiplying $V_1(x)V_2(x)$ by the square of (14) and integrating. Dominated convergence shows this also to be $o(1)$. This completes the proof of the lemma.

4 Completion of the proof

Recall that A_n is the Laguerre kernel (4) and B_n is the compressed Bessel kernel (5), which we also denote in its operator version as $M_V\, J_n\, M_V$. We shall show first that

$$\det(I + A_n) \sim \det(I + B_n)$$

as $n \to \infty$. (It is clear that A_n is a finite rank operator and so it is trace class, and using the integral representation (8) for the compressed Bessel kernel and integrating over t shows that B_n is also trace class. Thus both determinants are defined.) This will follow from what we have already done once we know that the operators $I + B_n$ are uniformly invertible, which means that they are invertible for sufficiently large n and the operator norms of their inverses are $O(1)$.

Lemma 4.1 *The operators $I + B_n$ are uniformly invertible and*

$$(I + B_n)^{-1} = I - M_V J_n M_{VU^{-1}} + O_2(1). \tag{15}$$

Proof. We replace the operator by its unitary equivalent $M_{\tilde{V}}\, J_n\, M_{\tilde{V}}$ where now J_n is given by (12), or equivalently

$$J_n(x,y) = \sqrt{xy}\int_0^{2\sqrt{n}} J_\nu(xt)\, J_\nu(yt)\, tdt$$

and we set $\tilde{V}(x) = V(x^2)$. If we set $H(x,y) = \sqrt{xy} J_\nu(xy)$ with H the corresponding operator (the Hankel transform), and denote now by P_n multiplication by the characteristic function of $(0,\, 2\sqrt{n})$, then $J_n = HP_nH$ and so $I + B_n$ is unitarily equivalent to $I + M_{\tilde{V}} HP_nHM_{\tilde{V}}$. These operators will be uniformly invertible if $I + P_nHM_{\tilde{V}^2}HP_n$ are.

Now it has recently been shown [4] that the operator $HM_{\tilde{V}^2}H$ is of the form $W(\tilde{V}^2) + K$, where $W(\tilde{V}^2)$ denotes the Wiener-Hopf operator with symbol $\tilde{V}(x)^2 = V(x^2)^2$ and K is a compact operator on $L^2(\mathbf{R}^+)$. (Much less is needed for this than that $\tilde{V}^2$ be a Schwartz function.) Since $1 + V(x^2)^2 = U(x^2)$ is nonzero and, being even, has zero winding number it follows from general facts about truncations of Wiener-Hopf operators that the operators $I + P_nW(\tilde{V}^2)P_n$ are uniformly invertible. Then since K is compact it follows that the $I + P_n(W(\tilde{V}^2) + K)P_n$ will be uniformly invertible if the limiting operator $I + W(\tilde{V}^2) + K = I + HM_{\tilde{V}^2}H$ is invertible. (For an exposition of the facts we used here see, for example, Chap.2 of [5].) However, since $H^2 = I$ the inverse of $I + H\, M_{\tilde{V}^2}\, H$ is easily seen to be $I + H\,W\,H$ where $W = -(1 + \tilde{V}^2)^{-1}\tilde{V}^2$. This establishes the first statement of the lemma.

For the second statement we apply Lemma 3.6 and use the facts $J_n^2 = J_n$ (which follows from $H^2 = I$) and that V^2 is a Schwartz function to see that for any bounded function W we have

$$(I + B_n)(I + M_V J_n M_W) = (I + M_V J_n M_V)(I + M_V J_n M_W)$$

$$= I + M_V J_n M_{V+V^2W+W} + O_2(1) = I + M_V J_n M_{V+UW} + O_2(1).$$

If we choose $W = -VU^{-1}$ and multiply both sides by $(I + B_n)^{-1}$ we obtain the result.

Lemma 4.2 $\det(I + A_n) \sim \det(I + B_n)$ *as* $n \to \infty$.

Proof. If an operator C is trace class then

$$\det(I + C) = \tilde{\det}(I + C)e^{-\mathrm{tr}\, C}$$

where $\tilde{\det}$ is the generalized determinant [7]. (The generalized determinant is defined for any Hilbert-Schmidt operator.) Hence we can write

$$\frac{\det(I + A_n)}{\det(I + B_n)} = \det((I + A_n)(I + B_n)^{-1})$$

$$= \det(I + (A_n - B_n)(I + B_n)^{-1}) = \tilde{\det}(I + (A_n - B_n)(I + B_n)^{-1})\, e^{-\mathrm{tr}\,(A_n - B_n)(I + B_n)^{-1}}.$$

It follows from Lemmas 3.4(ii) and 3.5 that $A_n - B_n \to 0$ in Hilbert-Schmidt norm, and therefore from the uniform invertibility of the $I + B_n$ that the same is true of $(A_n - B_n)(I +$

$B_n)^{-1}$. Therefore from the continuity of the generalized determinant in Hilbert-Schmidt norm that we conclude that the generalized determinant above has limit 1. Thus it suffices to show that $\mathrm{tr}\,(A_n - B_n)(I + B_n)^{-1} \to 0$ as $n \to \infty$. Since $A_n - B_n \to 0$ in Hilbert-Schmidt the $O_2(1)$ term in (15) contributes $o(1)$ to the trace of the product. By Lemma 3.7 the term $M_V J_n M_{VU^{-1}}$ in (15) also contributes $o(1)$. That $\mathrm{tr}\,(A_n - B_n)$ itself is $o(1)$ follows easily from arguments already given—one can check that at each stage the traces of the error operators tend to zero.

Finally, we can quote the main result of [2] which gives the asymptotics of $\det\,(I + B_n)$ or, more exactly the determinants of their unitary equivalents. The formula is

$$\det(I + B_n) \sim \exp\{\frac{2n^{1/2}}{\pi}\int_0^\infty \log(U(x^2)dx - \nu/2 \log U(0) + \frac{1}{2\pi^2}\int_0^\infty xS(x)^2 dx\}$$

where $S(x) = \int_0^\infty \cos(xy) \log(U(y^2)dy$. This gives

Theorem 4.3 *Suppose U is nowhere zero and $U - 1$ is a Schwartz function. Then (1) holds.*

As mentioned in the introduction this result was computed heuristically in [3] using the Coulomb fluid approach. In the same paper the analogous result was also obtained for weights supported on the entire real line. These determinants involve Hermite polynomials. It is highly likely that the results here (and techniques) could also be extended to that case.

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